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1997 J. Phys. A: Math. Gen. 30 6737

(http://iopscience.iop.org/0305-4470/30/19/014)

J. Phys. A: Math. Gen. **30** (1997) 6737–6741. Printed in the UK PII: S0305-4470(97)84310-7

A property of the vacuum stress tensor in static Casimir theory

A A Actor† and I Bender‡

† Department of Physics, The Pennsylvania State University, Fogelsville, PA 18051, USA
‡ Institut für Hochenergiephysik der Universität Heidelberg, 69120 Heidelberg, Germany

Received 19 May 1997

Abstract. A quantum-field theory defined in any space having arbitrary static background structure along some axes but translation invariance along other *free* directions (labelled by i) is considered. This system has a renormalized vacuum-stress tensor $T_{\mu\nu}$ satisfying $T_{00} = -T_{ii}$. This relation has nothing to do with $T_{\mu\nu}$ being traceless or divergence free (neither property is assumed).

In the course of calculating [1, 2] the renormalized canonical and improved vacuum-stress tensors, $T_{\mu\nu}$ and $\Theta_{\mu\nu}$, respectively (i.e. the vacuum expectation values of the corresponding operator-valued tensors) for many Casimir systems which have translation invariance in some spatial directions (call the coordinates corresponding to these free directions q^i) but not in others (denoted by Q^i) we noted the evident existence of a general theorem for massive or massless scalar fields:

$$T_{00} = -T_{ii} \qquad \Theta_{00} = -\Theta_{ii} \tag{1}$$

where i labels all free directions. Numerous examples make clear the background structure along directions orthogonal to the free directions q^i has no influence on these statements. Hence they should be easily proven with unrestricted generality. We provide this proof here.

The relations (1) probably seem familiar. For example, in the classic calculation [3] of the vacuum-stress tensor $T_{\mu\nu}$ of the electromagnetic field between parallel metal plates (at $x_1 = 0, L$) $T_{\mu\nu}$ is traceless and conserved. From these properties and from the symmetries of the problem it follows that $T_{\mu\nu}$ must have the form [3]

$$T_{\mu\nu} = \text{constant}\left[\frac{1}{N+1}\eta_{\mu\nu} - n_\mu n_\nu\right] \qquad \eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1)$$

where N is the dimension of space and $n = (0, 1, 0, \ldots, 0)$ is the unit $(N+1)$ vector perpendicular to the metal plates. Thus $T_{00} = -T_{ii}$ for $i > 1$.

Equations (1) are substantially more general statements about tensors which may be more or less complicated functions of the non-free coordinates. For a scalar field $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are in general neither traceless nor divergence free. Nonetheless equations (1) hold and render many repetitious calculations unnecessary. In this paper we prove equations (1) directly using mode-sum formulae for $T_{\mu\nu}, \Theta_{\mu\nu}$. Then we observe that equations (1) have a general covariance basis. This simple theorem is thus seen to be valid for quite general quantum-field theories.

For a scalar quantum field $\hat{\phi}$ defined on any static space one can construct the components of $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ from the vacuum expectation value $\langle\hat{\phi}(x)\hat{\phi}(y)\rangle$ by performing the differentiations with respect to x and/or y appearing in the definition of $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ and then letting $x \to y$. This expectation value is most conveniently obtained as follows. In the presence of background structure $V(\boldsymbol{x})$ coupled bilinearly to $\hat{\phi}$ and including possible spatial curvature, $\hat{\phi}$ satisfies $[\Box + V(\boldsymbol{x})]\hat{\phi} = 0$ (in the massive case $V(\boldsymbol{x})$ is meant to include the mass term). Defining the modes $\phi_n(\boldsymbol{x})$ as the solutions of the eigenvalue equation $[-\triangle + V(\boldsymbol{x})]\phi_n = \omega_n^2\phi_n$ the field operator $\hat{\phi}(x_0, \boldsymbol{x})$ is expressible in terms of these and the eigenvalues ω_n in the familiar way

$$\hat{\phi}(x) = \sum_n \frac{1}{\sqrt{2\omega_n}}[\hat{a}_n \mathrm{e}^{-\mathrm{i}\omega_n x_0}\phi_n(\boldsymbol{x}) + \hat{a}_n^\dagger \mathrm{e}^{\mathrm{i}\omega_n x_0}\bar{\phi}_n(\boldsymbol{y})] \tag{2}$$

$\hat{a}_n, \hat{a}_n^\dagger$ fulfilling the usual commutation relations of annihilation and creation operators and $\bar{\phi}_n(\boldsymbol{x})$ being the complex conjugate of $\phi_n(\boldsymbol{x})$. Inserting this expression in $\langle\hat{\phi}(x)\hat{\phi}(y)\rangle$ one obtains

$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{1}{2}\sum_n \frac{1}{\omega_n}\mathrm{e}^{-\mathrm{i}\omega_n(x_0-y_0)}\phi_n(\boldsymbol{x})\bar{\phi}_n(\boldsymbol{y}). \tag{3}$$

(The sum on the right is, of course, to be understood symbolically, implying also integration over continuous parameters.)

Our only assumption will be that space is a direct product $\mathcal{M} \times E^f$ of some static manifold $\mathcal{M}$ which may be arbitrarily complicated (because it plays no role in the considerations to follow) and an f-dimensional free, boundaryless subspace E^f. The spatial coordinates $\boldsymbol{x}$ then separate into coordinates $\boldsymbol{Q}$ on $\mathcal{M}$ and coordinates $\boldsymbol{q} = (q^1, \ldots, q^f)$ (chosen Cartesian) on E^f: $\boldsymbol{x} = (\boldsymbol{Q}, \boldsymbol{q})$. The background potential $V(\boldsymbol{x}) = V(\boldsymbol{Q})$, if present, depends only on the non-free coordinates $\boldsymbol{Q}$. The modes factorize

$$\phi_n(\boldsymbol{x}) = \psi_n(\boldsymbol{Q})(2\pi)^{-f/2}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{q}}$$

where $\psi_n(\boldsymbol{Q})$ are solutions of the eigenvalue equation

$$[-\triangle_Q + V(\boldsymbol{Q})]\psi_n(\boldsymbol{Q}) = \lambda_n^2\psi_n(\boldsymbol{Q}) \qquad \text{and} \qquad \omega_n^2 = \lambda_n^2 + \boldsymbol{p}^2. \tag{4}$$

Consequently the heat kernel corresponding to the system under consideration factorizes as well:

$$\begin{aligned} K(t|\boldsymbol{x}, \boldsymbol{x}') &\equiv \sum_n \mathrm{e}^{-t\omega_n^2}\phi_n(\boldsymbol{x})\bar{\phi}_n(\boldsymbol{x}') \\ &= K(t|\boldsymbol{Q}, \boldsymbol{Q}')K(t|\boldsymbol{q}, \boldsymbol{q}')_{\text{free}} \end{aligned} \tag{5}$$

$$K(t|\boldsymbol{q}, \boldsymbol{q}')_{\text{free}} = (4\pi t)^{-f/2}\mathrm{e}^{-(\boldsymbol{q}-\boldsymbol{q}')^2/4t}. \tag{6}$$

The heat kernel $K(t|\boldsymbol{Q}, \boldsymbol{Q}')$ on $\mathcal{M}$ is constructed from $\psi_n(\boldsymbol{Q})$ and λ_n just as is $K(t|\boldsymbol{x}, \boldsymbol{x}')$ from ϕ_n and ω_n in equation (5). K_{free} in equation (6) is the f-dimensional free-space heat kernel.

Following standard analytic continuation methods K is Mellin-transformed to

$$\sum_n (\omega_n^2)^{-s}\phi_n(\boldsymbol{x})\bar{\phi}_n(\boldsymbol{x}') = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}K(t|\boldsymbol{x}, \boldsymbol{x}') \tag{7}$$

where s is complex. As initially defined equation (5) is *unrenormalized*. With $\boldsymbol{x} = \boldsymbol{x}'$ the mode sum on the left of equation (7) diverges (converges) for $\mathrm{Re}\, s \leqslant N/2$ ($> N/2$)

where—as above—N is the dimension of space. Correspondingly with $\boldsymbol{x} = \boldsymbol{x}'$ the integral over t on the right diverges (converges) for $\mathrm{Re}\, s \leqslant N/2$ ($> N/2$), due to a non-integrable singularity of the integrand at $t = 0$. At least for $M \neq 0$ there are no divergence problems at the upper end of the integration interval since in this case, because of $\omega_n^2 \geqslant M^2 > 0$, the exponential, $\exp(-t\omega_n^2)$, in the definition of K in (5) provides an exponential cut-off of the integrand for $t \to \infty$.

Ultraviolet renormalization consists of subtracting the leading divergent terms from $K(t|\boldsymbol{x}, \boldsymbol{x}')$, and thereby the leading divergences from the mode sum on the left. There is a traditional way of doing this. For $\boldsymbol{x} = \boldsymbol{x}'$ the asymptotic heat kernel expansion (see e.g. [4, 5])

$$K(t|\boldsymbol{x}, \boldsymbol{x}) \sim a_0 t^{-N/2} + a_1 t^{-(N-1)/2} + \cdots \qquad t \to 0_+$$

identifies the divergent terms in equation (7). These terms can then be subtracted—as many as need to be—to define the *renormalized* heat kernel K_R from which we obtain the *renormalized* mode sum

$$\sum_n (\omega_n^2)^{-s} \phi_n(\boldsymbol{x}) \bar{\phi}_n(\boldsymbol{x}')|_R \equiv \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} K(t|\boldsymbol{x}, \boldsymbol{x}')_R. \tag{8}$$

All we need here from this general theory is that it exists. The integral (8) is by construction convergent for the range of s in which we use it.

Now set $\boldsymbol{Q} = \boldsymbol{Q}'$ and insert equation (5) into equation (8)

$$\sum_n (\omega_n^2)^{-s} \phi_n(\boldsymbol{x}) \bar{\phi}_n(\boldsymbol{x}')|_R = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} K(t|\boldsymbol{Q}, \boldsymbol{Q})_R (4\pi t)^{-f/2} \mathrm{e}^{-(q-q')^2/4t}. \tag{9}$$

Because $K(t|\boldsymbol{q}, \boldsymbol{q})_{\text{free}} = (4\pi t)^{-f/2}$ is merely a power of t, the subtraction procedure transfers smoothly from $K(t|\boldsymbol{x}, \boldsymbol{x}')$ to $K(t|\boldsymbol{Q}, \boldsymbol{Q}')$. Operating on equation (9) with $(\partial/\partial q^i)(\partial/\partial q'^i)$, $(\partial/\partial q^i)^2$ and $(\partial/\partial q'^i)^2$ and then setting $\boldsymbol{q} = \boldsymbol{q}'$ simply generates integrand factors $1/(2t)$, $-1/(2t)$ and $-1/(2t)$ respectively in equation (9). Thus the renormalized mode sums satisfy (in the s range where we use them)

$$\begin{aligned}
\sum_n (\omega_n^2)^{-s} |\partial_i \phi_n|_R^2 &= -\sum_n (\omega_n^2)^{-s} \phi_n \partial_i^2 \bar{\phi}_{nR} \\
&= -\sum_n (\omega_n^2)^{-s} (\partial_i^2 \phi_n) \bar{\phi}_{nR} \\
&= \frac{\Gamma(s-1)}{2\Gamma(s)} \sum_n (\omega_n^2)^{1-s} |\phi_n|_R^2.
\end{aligned} \tag{10}$$

The thing to do now is write down the mode sums representing $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ obtained from equation (3). These can be found in [1] and elsewhere in the literature of course. For completeness we give the necessary formulae again here:

$$T_{00} = \frac{1}{4} \sum_n \left(\omega_n + \frac{M^2}{\omega_n} \right) |\phi_n|^2 + \frac{1}{4} \sum_n \frac{1}{\omega_n} |\boldsymbol{\nabla} \phi_n|^2 \tag{11}$$

$$T_{aa} = \frac{1}{2} \sum_n \frac{1}{\omega_n} |\partial_a \phi_n|^2 - \frac{1}{4} \sum_n \frac{1}{\omega_n} |\boldsymbol{\nabla} \phi_n|^2 + \frac{1}{4} \sum_n \left(\omega_n - \frac{M^2}{\omega_n} \right) |\phi_n|^2$$
$$a = 1, 2, \ldots, N \tag{12}$$

$$\Theta_{00} = \frac{2N-1}{4N} \sum_n \omega_n |\phi_n|^2 + \frac{1}{4} \sum_n \frac{1}{\omega_n} |\boldsymbol{\nabla} \phi_n|^2 + \frac{M^2}{4N} \sum_n \frac{1}{\omega_n} |\phi_n|^2 \tag{13}$$

$$\Theta_{aa} = \frac{N+1}{4N}\sum_n \frac{1}{\omega_n}|\partial_a\phi_n|^2 + \frac{1}{4N}\sum_n \omega_n|\phi_n|^2 - \frac{1}{4N}\sum_n \frac{1}{\omega_n}|\nabla\phi_n|^2 - \frac{M^2}{4N}\sum_n \frac{1}{\omega_n}|\phi_n|^2$$
$$-\frac{N-1}{8N}\sum_n \frac{1}{\omega_n}[\phi_n\partial_a^2\bar{\phi}_n + (\partial_a^2\phi_n)\bar{\phi}_n] \qquad a = 1, 2, \ldots, N. \tag{14}$$

Note that equations (12) and (14) hold for all spatial dimensions, not only the free ones. The proof of equation (1) can now be completed.

Equation (10) shows that

$$\sum_n \frac{1}{\omega_n}|\partial_i\phi_n|_R^2 = -\sum_n \frac{1}{\omega_n}\phi_n(\partial_i^2\bar{\phi}_n)_R$$
$$= -\sum_n \frac{1}{\omega_n}(\partial_i^2\phi_n)\bar{\phi}_{nR} = -\sum_n \omega_n|\phi_n|_R^2 \tag{15}$$

where ∂_i means differentiation with respect to q^i, q'^i $(i = 1, \ldots, f)$. Then equations (1) follow at once from equations (11)–(14). $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are functions only of coordinates $\boldsymbol{Q}$ of course.

It is easy to show [1] that $T_{ij} = 0, \Theta_{ij} = 0$ for $i \neq j$. Thus equations (1) really tell us that

$$T_{AB} = \eta_{AB}T(\boldsymbol{Q}) \qquad \Theta_{AB} = \eta_{AB}\Theta(\boldsymbol{Q}) \tag{16}$$

where A, B are indices running over $0, 1, 2, \ldots, f$. Here we are introducing an $(f+1)$-dimensional Minkowski sub-spacetime M^{f+1} with coordinates $x^A = (x^0, \boldsymbol{q})$. In M^{f+1} the subtensors of the full tensors $T_{\mu\nu}, \Theta_{\mu\nu}$ displayed in equations (16) have the covariant form shown. The scalar (under Lorentz transformations in M^{f+1}) functions T, Θ depend on the non-free spatial coordinates $\boldsymbol{Q}$. These functions are analogous to cosmological constants in M^{f+1} (of course they are only independent of the coordinates x^A on M^{f+1}). The subtensors (16) are conserved in M^{f+1}

$$\partial^B T_{AB} = 0 \qquad \partial^B\Theta_{AB} = 0. \tag{17}$$

If we had been more clever could we have predicted equations (16)? Yes—for the same reason one can predict $T_{\mu\nu} = \eta_{\mu\nu}\Lambda$ in Minkowski spacetime in the complete absence of background spatial structure. There are no preferred directions in M^{f+1} and the only tensor available for the construction of T_{AB}, Θ_{AB} is η_{AB}. This is, of course, the easiest path to equation (16). Still, it is perhaps not a bad thing to have an explicit derivation, if only to verify that the field-theory machinery is working properly. (Verification of equation (16) in other field theories could similarly be a useful check—see below.)

Once one has reached this point it becomes obvious that equation (16) holds at a level far more general than the single-scalar-field context in which we first derived it. Consider a relativistic field theory (including fields with spin and interacting fields) defined on Minkowski spacetime M^{N+1}. Imagine redefining this theory on spacetime $M^{f+1} \times \mathcal{M}$ (with the same dimension $N+1$ of course) to create a static Casimir effect of arbitrary complexity with $\mathcal{M}$ deforming (away from spatial uniformity) the quantum fields of the original theory. The subspace $\mathcal{M}$ contains background spatial structure (boundaries, topology, local curvature, etc) which interacts directly with some or all of the fields of the theory. Nonetheless in M^{f+1} there are no preferred directions. Clearly, as long as η_{AB} remains the only tensor available in M^{f+1} for the construction of the M^{f+1} subtensors T_{AB}, Θ_{AB} we can expect equations (16) and (17) to hold.

It would be surprising if such a general and simple theorem has not previously been noticed. Yet we never see it used in work on Casimir theory or in related areas of quantum field theory. Being a real labour saver this theorem deserves to be widely known.

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